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Stability of bound states of pulses in the Ginzburg-Landau equations

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We consider bound states of quasisoliton pulses in the quintic Ginzburg-Landau equation and in the driven damped nonlinear Schrödinger equation. Using the perturbation theory, we derive dynamical systems describing the interaction between weakly overlapping pulses in both models. Bound states (BS's) of the pulses correspond to fixed points (FP's) of the dynamical system. We found that all the FP's in the quintic model are unstable due to the fact that the corresponding dynamical system proves to have one *negative* effective mass. Nevertheless, one type of FP, spirals, has an extremely weak instability and may be treated in applications as representing practically stable BS's of the pulses. If one considers an extremely long evolution, the spiral gives rise to a stable dynamical state in the form of an infinite-period limit cycle. For the driven damped model, we demonstrate the existence of fully stable BS's, provided that the amplitude of the driving field exceeds a very low threshold. [S1063-651X(97)02510-5]

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I. INTRODUCTION

Various forms of the Ginzburg-Landau (GL) equations and solitary-pulse (SP) solutions to them is a topic that has been attracting a great deal of attention; see, e.g., [1–16]. This is stimulated both by physical applications, which extend from nonlinear fiber optics to traveling-wave convection in binary fluids, and by the interest to fundamental dynamical properties of models based on the GL equations.

After finding stable SP's, the next natural step is to consider their interactions (see, e.g., [17]). Here an issue of obvious importance is the possibility of the existence of bound states (BS's) of the pulses. A general idea for the formation of the BS's was put forth in [7,10]: A combination of the conservative (dispersion) and dissipative (gain and/or losses, including diffusionlike losses) terms, characteristic for the GL equations, renders the ''tails'' of the pulses exponentially decaying with oscillations rather than simply decaying. If one may introduce an effective potential of interaction between the SP's [18,19] (or the so-called pseudopotential for a strongly dissipative model [7,12], which is related to the model's Lyapunov's function the same way as the usual potential relates to the Hamiltonian of the nearly conservative model), the oscillating tails will naturally give rise to local minima of the interaction potential, which in turn may account for locally stable BS's.

Some evidence of the BS existence has been found in earlier numerical experiments (see, e.g., [6]). In a paper entirely devoted to the BS, simulations allowed one to find them directly [20]. However, this paper was limited to in-phase and π out-of-phase solitons only. Next, the phase difference between solitons was varied, and it was found that the BS's of the type predicted in [7] by means of the perturbation theory are unstable [21]. Namely, all BS's with the phase difference 0 or π between the solitons are of saddle type, so if the BS is stable to the perturbations of the separation between the solitons, it is unstable to the perturbations of the phase and vice versa (see also Refs. [22,23]). However, these results were obtained for a strongly perturbed system (because in the case of really small parameters, it was very difficult to generate BS's numerically) and only for one particular set of parameters. Moreover, the in-phase BS's were not considered at all in the negative-dispersion model. Nevertheless, there was an apparent contradiction between these results and the predictions of [7,10], as well as the earlier simulations [6] that showed stable BS's.

A different but related model, viz., the driven-damped nonlinear Schrödinger (NLS) equation, was introduced by Kaup and Newell [24–27]. This model also has a SP solution with oscillating tails and two such SP's can potentially form a BS. Numerical results specially aimed at study of the BS's in this model were reported in [28] (see also [26,29]). It was found in [28] that the BS's exist indeed and the separation between pulses is in fairly good agreement with the analytical predictions, even in the case when the parameters assumed to be small in the analysis were actually not so small. However, the numerical study was limited to in-phase solitons and no phase perturbations have been discussed. Also, the text of the paper contains some inaccuracies [30].

Thus there is a certain lack of clear understanding of fundamental properties of the BS's, especially as concerns their stability. The objective of the present paper is to advance understanding of these crucial properties of the BS's. We will consider them in two most important models, viz., the quintic GL and driven damped NLS equations, in the case when they may be treated as perturbations of the NLS equation. This case is quite realistic for the applications (at least) to the optical fibers and, simultaneously, it admits a consistent analysis based on the perturbation theory. We will derive a system of ordinary differential equations governing the interaction of two weakly overlapping SP's, look for fixed points (FP's), and study their stability. The most essential results will be that in the driven damped NLS equation the BS's are stable, while in the quintic model they are unstable,

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but the instability of some of them (with the phase difference $\pi/2$ between the bound pulses) is extremely weak. Moreover, we will demonstrate that those weakly unstable stationary BS's give rise to truly stable dynamical states in which the two pulses remain bound together.

II. THE QUINTIC GINZBURG-LANDAU MODEL

We take the quintic GL equation in the standard form that emphasizes its proximity to the unperturbed NLS limit:

$$iu_z+\frac{1}{2}u_{\tau\tau}+|u|^2u=-i\alpha u+i\beta u_{\tau\tau}+i\gamma|u|^2u-i\Gamma|u|^4u. \tag{1}$$

Here we are using the ''fiber'' notation, i.e., z and τ are the propagation distance and the so-called reduced time. All the parameters α, β, γ, and Γ are assumed to be positive. They account for, respectively, the linear losses, spectral filtering (or diffusion in other physical contexts), nonlinear gain, and stabilizing higher-order nonlinear losses.

The SP solutions are assumed to be close to the soliton of NLS with an amplitude η,

$$u=\eta\ \mathrm{sech}[\eta(\tau-T)]e^{i[(1/2)\eta^2z+\phi]}, \tag{2}$$

T and ϕ being arbitrary constants. The perturbation theory can be applied provided that the dimensionless parameters β, γ, and $\alpha\Gamma$ are all small. If this is the case, the amplitude is [3]

$$\eta^2=(16\Gamma)^{-1}[5(2\gamma-\beta)\pm\sqrt{25(2\gamma-\beta)^2-480\alpha\Gamma}], \tag{3}$$

where the upper and lower signs correspond, respectively, to stable and unstable pulses (see also Refs. [31, 32] for a comprehensive discussion of stability). In what follows, we choose the upper sign since we are interested in the SP's that are by themselves stable. In addition to selecting the definite value of the soliton's amplitude, the small dissipative perturbations also make the asymptotic form of the soliton far from its center oscillating:

$$u\approx 2\eta e^{-\eta|\tau|+i\chi|\tau|}, \tag{4}$$

where $\chi=\alpha\eta^{-1}+\beta\eta$.

The next step is to consider the interaction between two weakly overlapping pulses with equal amplitudes. We will introduce the normalized propagation distance $x\equiv 2\sqrt{2}\eta^2z$, the normalized separation between the SP's $r\equiv\eta(T_1-T_2)$, and the phase difference between them $\psi\equiv\psi_1-\psi_2$, where the subscripts 1 and 2 mark the positions and phase of the two interacting pulses (2). Regarding the overlapping between the solitons as another small perturbation, one can derive a system of effective evolution equations for r and ψ either by means of the direct perturbation-theory technique of Karpman and Solov'ev [19] (see also [33]) or using the variational approach [34]. The final form of the equations proves to be exactly the same:

$$\frac{d^2r}{dx^2}+\frac{\sqrt{2}}{3}\beta\frac{dr}{dx}+e^{-r}[\cos(br)+b\ \sin(br)]\cos\psi=0\ , \tag{5}$$

$$\frac{d^2\psi}{dx^2}+\frac{\lambda}{\sqrt{2}}\frac{d\psi}{dx}-e^{-r}\cos(br)\sin\psi=0, \tag{6}$$

where the four original control parameters combine into three final ones: β and

$$\lambda\equiv\frac{1}{15}\sqrt{25(2\gamma-\beta)^2-480\alpha\Gamma},\quad b\equiv\frac{-\alpha+\beta\eta^2}{\eta^2}. \tag{7}$$

Notice that $\cos(br)$ and $\sin(br)$ in Eqs. (5) and (6) are induced by the oscillations in the soliton's tail (4).

Equations (5) and (6) may be regarded as equations of motion for a mechanical system with two degrees of freedom, in the presence of friction, in the potential $U(r,\psi)=-e^{-r}\cos(br)\cos\psi$, which has a set of local extrema at

$$br_0=\tan^{-1}b+\frac{\pi}{2}(1+2n),\quad \psi_0=\pi m, \tag{8}$$

where $n=0,1,2,\ldots,\ m=\pm1,\pm2,\ldots$. In the case of a ''normal'' dynamical system, those extrema that provide for minima of the potential would be stable FP's of the underlying dynamical system and thus they would produce stable BS's of the two pulses [7]. However, a peculiarity of the system (5) and (6) is that, while the effective mass corresponding to the degree of freedom r is $+1$, that for ψ is -1. The negative effective mass drastically changes the stability of the FP's. In particular, all the local extrema (8) are *saddles* because of this. It is easy to find the pair of the eigenvalues that determine the character of the saddle FP:

$$\sigma_{1,2}=\pm b\sqrt{3/\beta\lambda}e^{-r_0}. \tag{9}$$

Due to the assumed smallness of the parameters on the right-hand side of Eq. (1), the coordinate r_0 of the FP given by Eq. (8) is large and hence the eigenvalues (9) are *exponentially* small. Notice that, in the framework of the fourth-order system (5) and (6), the FP must have four eigenvalues. However, two of them that are missing in Eq. (9) are negative and are not exponentially small, i.e., they correspond to quickly decaying (stable) small perturbations around the FP.

Besides the saddles (8), Eqs. (5) and (6) also have the second set of the FP's,

$$br_0=\frac{\pi}{2}(1+2n),\quad \psi_0=\frac{\pi}{2}(1+2m). \tag{10}$$

Comparing the FP's (8) and (10), we notice that, for coinciding values of the integer n, they have nearly equal separation r between the bound pulses, but the relative phase ψ differs by $\pi/2$. The stability analysis of the FP's (10) reveals that there are two relatively large negative eigenvalues corresponding to rapidly decaying perturbations [as well as in the case of the FP (8)] and two exponentially small complex eigenvalues

$$\sigma_{1,2}=\pm ib\sqrt{3/\beta\lambda}e^{-r_0}+\frac{3}{2}(b/\beta\lambda)^2\left(\sqrt{2}\beta+\frac{3}{\sqrt{2}}\lambda\right)e^{-2r_0}. \tag{11}$$

Obviously, the FP (10) is an unstable spiral. Thus we obtain two types of unstable BS's in the quintic GL model: Depending on the phase difference between the SP's, their BS should be unstable as a saddle or as a spiral. Exactly this was observed in the recent numerical experiments performed at *nonsmall* values of the perturbation parameters (and for the opposite sign in front of the dispersion term) [21]. Therefore, we conjecture that the results should plausibly remain valid even when the perturbation theory cannot be applied.

Returning to the perturbative analysis, we notice a very important difference of Eq. (11) from Eq. (9). Namely, for the same n, i.e., nearly the same r_0, the real part of the eigenvalue (11), accounting for the instability of the spiral, is proportional to the square of the exponentially small factor e^{-r_0}, while in the case of the saddle the instability growth rate was linear in this factor. Thus the instability of the spiral is extremely weak and one may interpret this FP, provided that the underlying perturbation parameters are small indeed (which is most frequently the case for applications to the optical fibers), as a practically stable BS of the pulses. This result should be amenable to experimental verification in the nonlinear optical fibers.

Despite the very weak instability of the spiral FP, it is a question of a fundamental interest to explore a result of the development of the instability at extremely large propagation distances. To this end, one can notice that the fourth-order system (5) and (6) implies relatively quick decay of the perturbations corresponding to the above-mentioned relatively large (non-exponential) stable eigenvalues, and a very slow evolution corresponding to the exponentially small eigenvalues (9) and (11). In this connection, a natural simplification of the full system will be to derive its projection on the two-dimensional space of the slow modes, eliminating the two rapidly decaying ones. Technically, this implies treating the second derivatives in Eqs. (5) and (6) as small perturbations. In the zeroth approximation, one simply omits the second derivatives, so that, Eqs. (5) and (6) reduce to

$$\frac{dr}{dx}=-\frac{3}{\sqrt{2}\beta}e^{-r}[\cos(br)+b\sin(br)]\cos\psi, \tag{12}$$

$$\frac{d\psi}{dx}=\frac{\sqrt{2}}{\lambda}e^{-r}\cos(br)\sin\psi. \tag{13}$$

Notice that, within the framework of this system, the FP (8) remains the saddle, while Eq. (10) is neutrally stable (i.e., it is the so-called center on the phase plane, surrounded by a family of closed trajectories).

At the next step, one restores the second-derivative term by means of the identity $d^2\psi/dx^2\equiv(d/dx)(d\psi/dx)$, and similarly for r, substituting for $d\psi/dx$ and dr/dx Eqs. (13) and (12). To perform the second differentiation, one uses Eqs. (12) and (13) once again. This procedure produces a number of terms which should be sorted out: some of them are unimportant corrections to the terms already present in Eqs. (12) and (13), while others are important, although exponentially small, accounting for, e.g., the weak instability of the spiral. Keeping the essential corrections, one eventually arrives at the simplified second-order system sought:

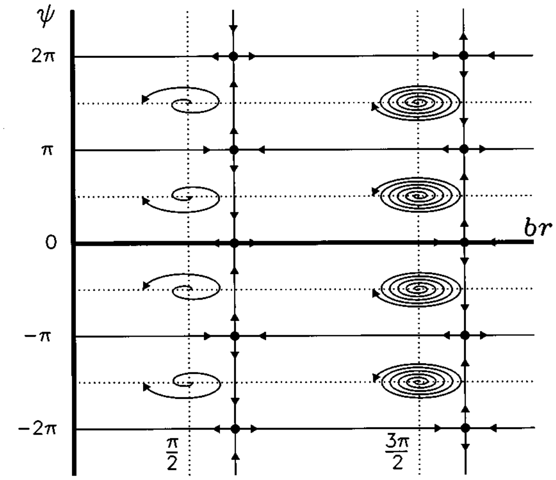

FIG. 1. Phase portrait of the reduced dynamical system (12) and (13).

$$\frac{dr}{dx}=-\frac{3}{\sqrt{2}\beta}e^{-r}[\cos(br)+b\sin(br)]\times\left[\cos\psi+\frac{3}{\beta\lambda}e^{-r}\cos(br)\sin^2\psi\right], \tag{14}$$

$$\frac{d\psi}{dx}=\frac{\sqrt{2}}{\lambda}e^{-r}\cos(br)\sin\psi-\frac{3}{\sqrt{2}\lambda^2\beta}be^{-2r}\times[\cos(br)+b\sin(br)]\sin(br)\sin(2\psi). \tag{15}$$

It is straightforward to verify that the reduced two-dimensional system (14) and (15) has exactly the same FP's (8) and (10) as the underlying four-dimensional system (5) and (6), with the FP's eigenvalues given by the same expressions (9) and (11). However, it is very easy now to understand the general character of the dynamical trajectories on the phase plane of the reduced system, without any actual computations. Indeed, one can immediately check that the saddles (8) are connected by a rectangular grid of special trajectories of the form $r\equiv r_0$, $\psi=\psi(x)$ and $r=r(x)$, $\psi\equiv\psi_0$, where r_0 and ψ_0 are the values at the FP's (8). These trajectories are stable and unstable separatrices of the saddles and they exist as exact solutions to Eqs. (5) and (6) and Eqs. (14) and (15). From this fact and our knowledge of the eigenvalues of the FP's, a qualitative phase portrait of the reduced system follows immediately, as shown in Fig. 1. Looking at Fig. 1, one concludes that the spirals, except for those corresponding to $n=0$ in Eq. (10), give rise, at $x\rightarrow\infty$, to infinite-period limit cycles coinciding with an elementary cell of the separatrix grid. The spirals corresponding to $n=0$, i.e., to the BS with the smallest possible separation between the SP's, formally give rise to a similar cycle, which, however, having a side at $r=0$, implies a collision between the two pulses. The latter event is not described by the above approximation.

To check the correctness of this picture, we performed numerical simulations of the system (14) and (15). The simulations produced results exactly complying with the picture displayed in Fig. 1 (that is why we do not show these nu-

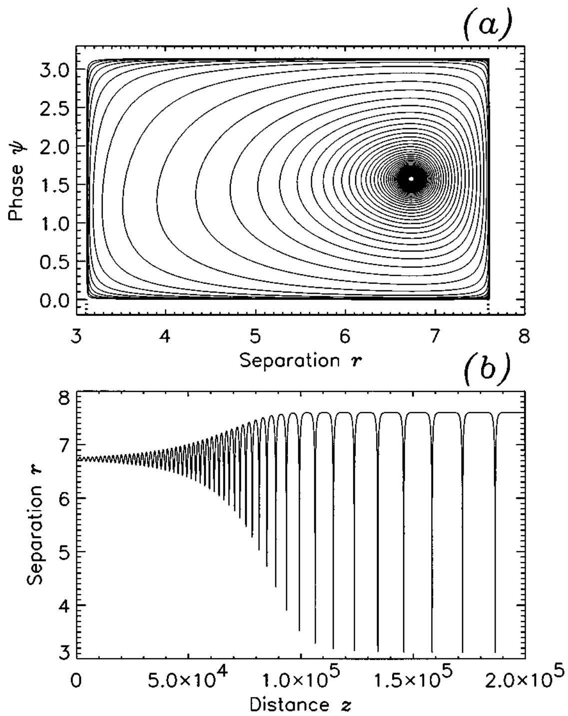

FIG. 2. Example of a dynamical trajectory of the full four-dimensional system (5) and (6) in projection onto the plane (r,ψ). The trajectory is unwinding around the fixed point (10) with $n=1$ (the ''hole'' is determined by the choice of the initial point). The parameters are $b=0.7$, $\beta=0.525$, and $\lambda=0.35$.

merical results: they do not convey any additional information). A more important issue for numerical verification is to simulate the full four-dimensional system (5) and (6) to see if its trajectories are indeed close to those of the reduced two-dimensional system. The result always was that they are very close indeed. As an illustration, in Fig. 2 we display a projection of the four-dimensional dynamical trajectory onto the plane (r,ψ). This trajectory pertains to the case $\lambda=\frac{1}{2}b$, $\beta=\frac{3}{4}b$ [in this case the dynamical system (5) and (6) coincides with that for the interacting SP's governed by the *cubic* GL equation, so this case is of additional interest), and $b=0.7$. The FP was taken as per Eq. (10) with $n=1$. Notice that the numerical values of the perturbation parameters are not really small in this case; nevertheless, the trajectory, exactly as it is predicted by the reduced system, is slowly unwinding around the FP, filling the interior of the separatrix-grid cell, and finally the motion practically stops when the trajectory gets very close to the boundaries of the cell.

Thus we arrive at a general conclusion that the BS's of the pulses described by this approximation may be either effectively stable in the usual sense, if one may neglect the exponentially weak instability, or stable as the dynamical states corresponding to the limit cycle. The former case most likely applies to usual solitons in the nonlinear optical fibers; the latter dynamical state should be observable in the optical-fiber experiments at extremely large propagation distances. Note that successful experiments demonstrating transmission of usual optical solitons over the distance of 10^6 km [35] suggest that observation of the stable dynamical state should be possible indeed. Alternatively, one can use shorter distances and narrower solitons, with the temporal width ~ 1 ps.

III. THE DRIVEN DAMPED MODEL

This model is based on the equation [24]

$$iu_t+\frac{1}{2}u_{xx}+|u|^2u=-i\alpha u+\epsilon e^{i\Omega t}, \tag{16}$$

where we have switched to the traditional (non-fiber-optics) notation, though this model has some optical applications too [26]. It is well known that this model supports two SP solutions (existing above a cw background supported by the drive in competition with the friction), one stable and one unstable [24,25]. Far from the center of the pulse, its asymptotic form is [cf. Eq. (4)]

$$u(x,t)\approx 2\eta e^{i\Omega t-\eta|x|+ik|x|+i\psi}, \tag{17}$$

where the soliton's amplitude η is related to the driving frequency by the relation $\eta=\sqrt{2\Omega}$ and ψ is a phase constant. The wave number in Eq. (17), because of which the soliton's tail is oscillatory and thus gives rise to an effective interaction potential with local minima, is $k=\alpha/\eta$ [7].

Combining the results of [19] and [7], it is straightforward to derive a system of equations describing the interaction of two weakly overlapping pulses in the model (16). An essential difference from the case considered in Sec. II is that we will have not two but three equations, as in this model not only the phase difference but also each phase by itself is nontrivial dynamical variable. The form of the equations simplifies in terms of the variables

$$2\sqrt{2}\eta^2 t\equiv\tau,\quad \eta\Delta\equiv r,\quad \alpha/\eta^2\equiv b,\quad \pi\epsilon/2\eta^3\equiv E, \tag{18}$$

where Δ is the separation between the centers of the two pulses. The eventual form of the dynamical system is

$$\frac{d^2\psi_j}{d\tau^2}+\sqrt{2}b\frac{d\psi_j}{d\tau}+\frac{1}{2}(-1)^{j-1}e^{-r}\cos(br)\sin(\psi_2-\psi_1)+\frac{1}{4}b+\frac{1}{4}E\ \sin\psi_j=0, \tag{19}$$

$$\frac{d^2r}{d\tau^2}+e^{-r}[\cos(br)+b\ \sin(br)]\cos(\psi_2-\psi_1)=0, \tag{20}$$

where j takes values 1 and 2, ψ_j being the phase constants of the two pulses.

The system (19) and (20) has the FP's

$$br_0=\frac{\pi}{2}+\tan^{-1}b+\pi n,\quad n=1,2,3,\ldots \tag{21}$$

$$\psi_1=\psi_2=-\sin^{-1}(b/E),\quad \psi_1=\psi_2=-\pi+\sin^{-1}(b/E), \tag{22}$$

which are similar to the FP's (8) considered in Sec. II. In what follows, the common values of $\psi_1=\psi_2$ at the FP will be denoted as ψ_0. Note that this FP exists if $|b/E|\leqslant 1$, which is a well-known threshold condition [24], which we will assume to be satisfied.

Stability analysis of the FP is straightforward. First of all, the soliton must be stable in isolation, which implies a well-known fact: Out of the two FP's in Eq. (22), one should take the one with $E\cos\psi_0>0$ [24]. Next, the perturbations of the separation and phase decouple in the linearized equations governing evolution of the small perturbations around the FP and in order to provide for its stability against the separation perturbations one should take $n=1$ in Eq. (21), which is known too [28]. After this, a remaining previously unexplored issue is an accurate analysis of the stability against phase perturbations. Technically, it is quite easy and leads to the final result: The phase perturbations do not produce instability provided that

$$\frac{1}{4}E^2\cos^2\psi_0>e^{-2r_0}. \tag{23}$$

The meaning of the condition (23) is quite obvious: Phase locking of both pulses to the external drive is able to suppress the phase instability that rendered the FP's (8), considered in Sec. II, unstable. In accord with this, condition (23) is not satisfied in the absence of the drive $(E=0)$, but if the drive is present, it is quite easy to satisfy this condition, as its right-hand side is exponentially small, while the left-hand side is not.

IV. CONCLUSION

In this work we have made an effort to clarify a practically important issue that has remained rather controversial, namely, stability of bound states of pulses in the quintic GL equation and in the driven damped NLS model, both of which are well known to support stable isolated pulses. Analyzing the case when dissipative coefficients in the equations are small, we have derived dynamical systems to govern interaction between two weakly overlapping pulses. The bound states are then represented by fixed points of those systems. Further analysis has demonstrated that all the fixed points of the system corresponding to the quintic equation are unstable. A fundamental cause for this instability is the fact that one of the two effective masses in this dynamical system is negative. Some of these fixed points (saddles) represent the bound states with a phase difference between the pulses being a multiple of π and are relatively strongly unstable. Other fixed points (spirals) represent the bound states whose phase difference is a semi-integer in units of π and their instability is extremely (exponentially) weak in comparison to that of the saddles, so that the corresponding bound states are *practically stable*. We have also analyzed the development of the weak instability of the spirals, concluding that it does not destroy the bound states even at indefinitely large propagation distances, but instead turns them into stable dynamical states, described by an infinite-period limit cycle in terms of the dynamical system. Asymptotically, this limit cycle coincides with an elementary cell of a network formed by separatrices of the saddles. These analytical results easily explain recent direct simulations of the pulse interaction in the quintic GL equation at *nonsmall* values of the dissipative parameters [20,21]. Observation of the dynamical states predicted in this work, viz., the limit cycle, remains a challenging problem for the direct partial differential equation simulations as well for a laboratory experiment with optical solitons.

In the driven damped model, the situation is essentially simpler. Using the description in terms of the dynamical system, we have demonstrated that the fixed point, corresponding to the pair of pulses stably locked to the driving force, can easily become stable, provided that the drive's amplitude exceeds a very low threshold value. This stable bound state was observed earlier in direct simulations of the driven damped model.

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